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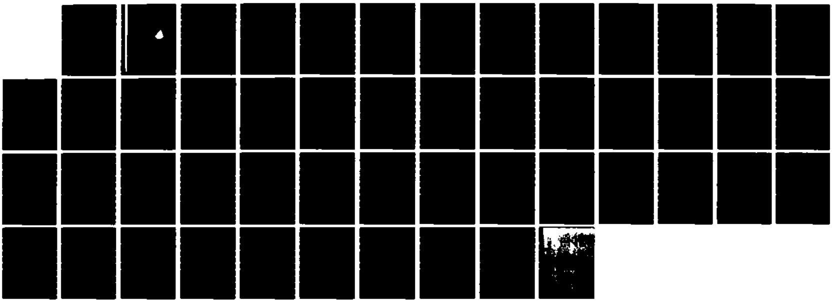

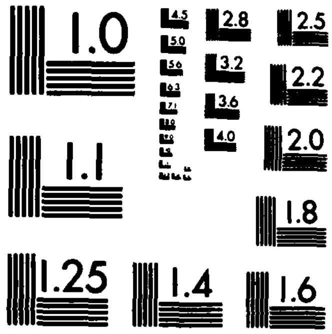

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by
CYRUS DERMAN, GERALD J. LIEBERMAN and SHELDON M. ROSS

OPERATIONS RESEARCH CENTER

UNIVERSITY OF CALIFORNIA · BERKELEY

ON THE USE OF REPLACEMENTS TO EXTEND SYSTEM LIFE

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ABSTRACT

A system has one vital component for which there are n spares. Whenever i failures of the vital component fails, the system fails. We are concerned with calculation and properties of the schedule of component replacements which maximizes the expected life of the system. Most of the paper deals with the case of $i = 1$.

ON THE USE OF REPLACEMENTS TO EXTEND SYSTEM LIFE

by

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1. Introduction

The literature dealing with the scheduling of replacements and/or inspections is very extensive. See [2] for a comprehensive set of references.

~~In this paper we are~~ This paper is concerned with the following question. A system has one vital component for which there are n spares. Whenever the vital component fails, the system fails. ~~We~~ The authors would like to schedule the replacement of the vital component with the spares so as to prolong the life of the system as much as possible. This problem can be generalized to where there are several components in the system and the scheduling of replacements refers to each of the components. ~~We~~ They deal mainly with the first question but treat, to some extent, a special case of the second. ~~In~~ The next to final section ~~we~~ considers a generalization which allows for the vital component to fail a fixed number of times before causing the system to fail.

2. One Vital Component Problem

Assume a one component system. Let F, with $\bar{F} = 1-F$, denote the life distribution of the component. We assume that its mean,

$$\mu = \int_0^\infty \bar{F}(t)dt < \infty ,$$

and that F has a continuous density f with support $(0,T)$, $0 < T \leq \infty$, and we let

$$r(t) = f(t)/\bar{F}(t)$$

denote its failure rate function.

Suppose the system is put into operation at $t = 0$, using an initial component. There are n statistically identical spares that can be used sequentially as replacements. The system fails whenever the component that is currently in use fails. The idea is to judiciously replace the component in use with an available spare in order to extend the life of the system. We assume that once a component has been removed it cannot be used again. Let S denote a replacement schedule, L the life of the system, and $E_S L$ the expected life of the system as a function of S. The problem of concern, here, is to determine the schedule S, and (or) some of its properties, that maximizes $E_S L$.

As will be evident there always exists an optimal schedule, although it may not be unique. Let v_n denote the expected life of the system given an optimal schedule of replacement of n spares. If $n = 0$, there is only one schedule; viz, the one that never replaces. Therefore,

$$v_0 = \mu \quad .$$

If $n \geq 1$, v_n satisfies the optimality equation

$$v_n = \max_{0 \leq x \leq T} \phi_n(x) \quad , \tag{1}$$

where

$$\phi_n(x) = \int_0^x tf(t)dt + \bar{F}(x)(x + v_{n-1}) \tag{2}$$

$$= \int_0^x \bar{F}(t)dt + \bar{F}(x)\, v_{n-1} \quad .$$

Since, by assumption, $\mu < \infty, \phi_n(x)$ is continuous at all $x \in [0,T]$. Thus, $\phi_n(x)$ can always be maximized, although not necessarily uniquely, at $x = x_n$ (say). The value, x_n, is, in words, the optimal length of time to use a component before replacing it when n spares are available. If $x_n = T$, then the component is never replaced and $v_n = \mu$. If $x_n = 0$, a component is immediately replaced, in which case $v_n = v_{n-1}$. It should be kept in mind that the system may fail before all n spaces are used even though a schedule is optimal.

Proposition 1: v_n is nondecreasing in n.

Proof: $v_n \geq \phi_n(0) = v_{n-1}$, $n \geq 1$.

Proposition 2: If $v_1 = v_0$, then $v_n = v_0$, $n \geq 1$.

Proof: From (1) and (2) we obtain, for $n \geq 2$,

$$(3) \qquad v_n - v_{n-1} = \phi_n(x_n) - \phi_{n-1}(x_{n-1})$$
$$\leq \phi_n(x_n) - \phi_{n-1}(x_n)$$
$$= \bar{F}(x_n)(v_{n-1} - v_{n-2}) \quad .$$

Since $v_{n-1} - v_{n-2} \geq 0$, by Proposition 1 and $0 \leq \bar{F}(x_n) \leq 1$, the proposition follows.

Since $\phi_1(0) = \phi_1(T) = \mu$, and from (1) and (2), it follows that $v_1 = v_0$ if and only if

$$(4) \qquad \mu \geq \int_0^x \bar{F}(t)dt + \bar{F}(x)\mu \quad , \quad 0 \leq x \leq T \quad ,$$

or, equivalently, if and only if,

$$\mu \leq \int_x^\infty \frac{\bar{F}(t)dt}{\bar{F}(x)} \quad , \quad 0 \leq x \leq T \quad .$$

The above condition is often called the new worse than used in expectation (NWUE) for it states that the expected additional life of a used vital component of age x is at least as large as the expected life of a new one.

By considering the derivative of the right side of (4) we can see that if $f(0) = r(0) < \frac{1}{\mu}$, then $v_1 > v_0$. On the other hand if F is NWUE then x_1 can equal T and $v_1 = v_0$. We shall assume for the remainder, unless stated otherwise, that F is not NWUE and so $v_1 > v_0$.

<u>Proposition 3</u>: $v_n > v_{n-1}$, $v_n - v_{n-1} < v_{n-1} - v_{n-2}$, and $0 < x_n \le x_{n-1} < T$, $n \ge 2$.

<u>Proof</u>: Since $v_1 > v_0$, $x_1 \ne 0$ or T. In a manner similar to showing (3) we have

$$v_n - v_{n-1} \ge \bar{F}(x_{n-1})(v_{n-1} - v_{n-2}) \quad , \quad n > 2 \quad . \tag{5}$$

If $x_n = T$ for some n, then $v_n = \mu = v_0$ which is contrary to proposition 1 in view of the hypothesis that $v_1 > v_0$. Therefore, $x_n < T$ for all $n \ge 1$, and, hence, $\bar{F}(x_n) > 0$. Thus, it follows inductively from (5) that $v_n > v_{n-1}$, $n \ge 1$. If $x_n = 0$ for some $n \ge 1$, then $v_n = v_{n-1}$, a contradiction of what has just been shown. Thus, $x_n > 0$, $\bar{F}(x_n) < 1$, $n \ge 1$. From (3) and the positivity of $v_n - v_{n-1}$, it follows that $v_n - v_{n-1} < v_{n-1} - v_{n-2}$, $n \ge 2$. Finally, from (3) and (5) and what has just been shown, we have

$$\bar{F}(x_n) \ge \bar{F}(x_{n-1}) \quad , \quad n \ge 2 \quad ,$$

from which $x_n \le x_{n-1}$, $n \ge 2$.

As previously remarked, x_n need not be unique. However, let $\{x_n\}$ denote the set of values for which the maximum of ϕ_n is obtained. The proof of Proposition 3 shows the following

<u>Corollary</u>: $\max \{x_n\} \le \min \{x_{n-1}\}$, $n \ge 2$.

Proposition 4: x_n satisfies

$$r(x_n) = \frac{1}{v_{n-1}} \quad , \quad n \geq 1 \quad ,$$

and r cannot be decreasing at x_n.

Proof: From proposition 3, x_n is an interior point of the interval $[0,T]$. On differentiating ϕ_n we get

$$\begin{aligned} \phi_n'(x) &= \bar{F}(x) - f(x)\, v_{n-1} \\ &= \bar{F}(x)(1 - r(x)\, v_{n-1}) \quad . \end{aligned}$$

Thus, $\phi_n'(x) = 0$ only if $r(x) = 1/v_{n-1}$. However, if $r(x) = \frac{1}{v_{n-1}}$ and x is a point of decrease of r, then ϕ_n' is increasing at x so that x cannot be a relative maximum.

Corollary: $x_n < x_{n-1}$, $n > 1$.

Proof: Proposition 3 asserts that $v_{n-1} > v_{n-2}$ and allows the possibility that $x_n = x_{n-1}$. However,

$$\begin{aligned} \frac{1}{v_{n-1}} &= r(x_n) \\ &= r(x_{n-1}) \\ &= 1/v_{n-2} \end{aligned}$$

is impossible.

The second derivative of ϕ_n'' is

$$\phi_n''(x) = -f(x) - f'(x)\, v_{n-1} \quad .$$

Thus, if x is a root of $r(x) = \frac{1}{v_{n-1}}$ and $f'(x) \geq 0$, then x is a relative maximum of ϕ_n.

If F is IFR, then x_n is the unique root of $r(x) = 1/v_{n-1}$. Because in this case, $f(0) = r(0) < 1/\mu$. In fact, from Proposition 3, $f(0) = r(0) < 1/v_n$, $n \geq 1$.

If r is unimodal and has two roots to $r(x) = 1/v_{n-1}$, then x_n is the smaller of the roots; if r has the so called "bathtub" shape, then x_n is the larger of the two roots.

<u>Proposition 5</u>: $\lim_{n\to\infty} v_n = v$ is finite or infinite according as $f(0)$ is positive or zero. Although x_n is not necessarily unique, $\lim_{n\to\infty} x_n = x^* \geq 0$, is a unique value.

<u>Proof</u>: If $v_1 = v_0$, then $v_n = \mu$, $n \geq 1$, by Proposition 2. Suppose, for the remainder of the proof, that $v_1 - v_0 > 0$. Since, by Proposition 1 $v_{n+1} \geq v_n$ then $\lim_{n\to\infty} v_n$ exists and is either finite or infinite. By Proposition 3 and the corollary following it, it is easily shown that $\lim_{n\to\infty} x_n$ exists and equals a unique non-negative value x^* (say). Suppose $f(0) = r(0) > 0$, $x^* = 0$, $\lim_{n\to\infty} v_n = \infty$. For $\varepsilon > 0$, choose $\delta > 0$ sufficiently small so that $r(x) \geq \varepsilon$ if $x \leq \delta$. If $v_N > 1/\varepsilon$, then $1/v < \varepsilon$, $\forall\, n \geq$ which implies that $x > \delta \;\forall\; n \geq N$, a contradiction

that $x^* = 0$. Thus, under these circumstances $\lim v_n = \infty$ is impossible. Suppose $f(0) = r(0) > 0$, $x^* > 0$. From Proposition 2

$$\begin{aligned} v_n - v_{n-1} &\le \bar{F}(x_n)(v_{n-1} - v_{n-2}) \\ &\le \bar{F}(x^*)(v_{n-1} - v_{n-2}) \\ &\le \bar{F}^{n-1}(x^*)(v_1 - v_0) \quad , \quad n \ge 2 \ . \end{aligned}$$

Thus

$$\begin{aligned} \lim_{n\to\infty} v_n - v_1 &\le \sum_{n=2} \bar{F}^{n-1}(x^*)(v_1 - v_0) \\ &= \frac{\bar{F}(x^*)}{F(x^*)}(v_1 - v_0) \\ &< \infty \ . \end{aligned}$$

Suppose, now, that $f(0) = 0$. Consider the schedule S having equal replacement intervals of length y with $n_y = [\frac{1}{F(y)}]$ equal to the number of available components, including the initial one. Let L denote the life of the system under S_{n_y}. Then it is easy to show that

$$\lim_{y\to 0} E(L) = \infty \ .$$

Since $v_{n_y} > E(L)$ and v_n increases in n it follows that $\lim_{n\to\infty} v_n = \infty$.

If $r(0) > 1/\mu$, then $x^* > 0$. For it is trivially true if $x_1 = T$; and if $x_1 < T$, then

$$
\begin{aligned}
r(x^*) &= \lim_{n\to\infty} r(x_n) \\
&= \lim_{n\to\infty} 1/v_{n-1} \\
&= 1/v \\
&< 1/v_0 \\
&= 1/\mu \quad .
\end{aligned}
$$

The following is a sufficient condition for x^* to equal 0.

Proposition 6: If F is a strictly new better than used (NBU(S)) distribution in the sense that

$$\frac{\bar{F}(t+x)}{\bar{F}(x)} < \bar{F}(t), \quad \text{for all} \quad x > 0 \quad , \quad t > 0 \quad ,$$

then $x^* = 0$.

Proof: Suppose F is NBU(S) and $x^* > 0$. For each n let S_n be the schedule $(x_{n/2}, x_{n/2}, x_{n-2}, \ldots, x_1)$; i.e., the first two replacement intervals of S_n are of length $x_{n/2}$ and optimal thereafter. Let

$$
\begin{aligned}
\bar{G}_n(t) &= \bar{F}(t) & , \quad t \leq x_{n/2} \\
&= \bar{F}(x_{n/2})\bar{F}(t-x_{n/2}) & , \quad t > x_{n/2} \quad .
\end{aligned}
$$

Note that

$$\bar{G}_n(t) > \bar{F}(t) \quad , \quad t > x_{n/2} \, .$$

Let $\tilde{v}_n = E_{S_n}(L)$. We have

$$\tilde{v}_n = \int_0^{x_n} \bar{G}_n(t)dt + G_n(x_n)v_{n-2} \quad .$$

Denoting $\bar{G} = \lim_{n\to\infty} G_n$ and letting $n \to \infty$, we have

$$\lim_{n\to\infty} \tilde{v}_n - v = \int_0^{x^*} (\bar{G}(t) - \bar{F}(t))dt + (\bar{G}(x^*) - \bar{F}(x^*))v$$

$$> 0 \quad ,$$

a contradiction that $\lim_{n\to x} v_n \leq v$.

<u>Remark</u>: The condition that F is NBU(S) is weaker than that F is strictly IFR, i.e., that $r(t)$ is strictly increasing.

<u>Examples</u>:

(i) Uniform (0,1). Here,

$$\begin{aligned} f(t) &= 1 \\ F(t) &= t \qquad , \quad \text{for} \quad 0 < t < 1 \quad , \\ r(t) &= 1/1-t \end{aligned}$$

$$v_0 = \mu = 1/2 \quad , \quad v_1 = 5/8 \quad .$$

Since, $f(0) > 1$, $v < \infty$. From Proposition 4

$$x_n = 1 - v_{n-1}$$

and

$$v_n = \frac{(1-v_{n-1})^2}{2} + v_{n-1}((1 - v_{n-1}) + v_{n-1})$$

$$= \frac{(1-v_{n-1})^2}{2} + v_{n-1}$$

$$= 1/2(1 + v_{n-1}^2) \quad , \quad n \geq 1 \quad .$$

Some values of x_n and v_n will appear in Table 1. Since F is NBU(S) we see that $x^* = 0$ and $v = \frac{1}{r(0)} = 1$.

(ii) Linear (0,1); specifically,

$$f(t) = 2t$$

$$F(t) = t^2 \quad , \quad \text{for} \quad 0 < t < 1$$

$$r(t) = \frac{2t}{1-t^2}$$

$$v_0 = \mu = 2/3 \quad .$$

Since, $f(0) = 0$, $v = \infty$, $x^* = 0$ and

$$r(x_n) = \frac{2x_n}{1-x_n^2}$$

$$= 1/v_{n-1} \quad ,$$

leading to the quadratic equation

$$x_n^2 + 2v_{n-1}\, x_n - 1 = 0 \quad ,$$

for which the only positive root is

$$x_n = -v_{n-1} + \sqrt{1+v_{n-1}^2} \quad .$$

Thus,

$$v_n = 2 \int_0^{x_n} t^2 \, dt + (1 - x_n^2)(x_n + v_{n-1})$$

$$= 2/3 \, x_n^3 + (1 - x_n^2)(x_n + v_{n-1}) \quad .$$

The values of x_n and v_n can be computed recursively, commencing with $v_0 = 2/3$.

(iii)

$$f(t) = 2(1 - t)$$

$$F(t) = 2t - t^2 \quad , \quad \text{for} \quad 0 < t < 1 \quad ,$$

$$r(t) = \frac{2}{1-t}$$

$$v_0 = \mu = 1/3 \quad .$$

Here, $f(0) = 2$; hence $v < \infty$. F is IFR with $r(0) \leq \frac{1}{\mu}$; thus

$$r(x_n) = \frac{2}{1-x_n}$$

$$= 1/v_{n-1} \quad ,$$

i.e.,

$$x_n = 1 - 2v_{n-1} \quad .$$

Also,

$$v_n = x_n^2(1 - 2/3 \, x_n) + (x_n - 1)^2(x_n + v_{n-1}) \quad .$$

The values of x_n and v_n can be calculated recursively. To find v, it is easier to express v in terms of x^* and solve for x^* in the latter equation after having let $n \to \infty$. Out of this procedure comes the equation

$$x^{*2}(1/2 - x^*/3) = 0$$

from which

$$x^* = 0$$

is the only acceptable solution. Thus,

$$v = 1/2 \quad .$$

(iv) Two exponentials in parallel.

Let

$$f(t) = 2e^{-t}(1 - e^{-t})$$

$$F(t) = (1 - e^{-t})^2 \quad , \quad 0 < t < \infty$$

$$r(t) = \frac{2e^{-t}(1-e^{-t})}{1-(1-e^{-t})^2}$$

$$v_0 = \mu = 3/2 \quad .$$

This case arises when the component comprising the system is a module consisting of two independent components, each component having an exponential distribution with mean one.

Since $f(0) = 0$, $\lim v_n = \infty$. The equation for obtaining x_n in terms of v_{n-1} is

$$r(x_n) = \frac{2e^{-x_n}\left(1-e^{-x_n}\right)}{1-\left(1-e^{-x_n}\right)^2}$$

$$= 1/v_{n-1} \quad .$$

Solving explicitly yields

$$x_n = \log \frac{2v_{n-1}-1}{2(v_{n-1}-1)} \quad .$$

Substituting this, in (2) yields, after some algebra,

$$v_n = 3/2 + \frac{2(v_{n-1}-1)^2}{2v_{n-1}-1} \quad , \quad v_0 = 3/2 \quad .$$

Numerical values for x_n and v_n appear in Table 2.

Since F is strictly IFR and thus NBU(S), $x^* = 0$ and $v = 1/r(0) = \infty$.

3. Equal Interval Replacement Schedules

In this section we restrict the class of schedules to those having replacement intervals of equal length. Let $\psi_n(y)$ denote the expected life of the system when y is the length of the interval. Let

$$\psi_n = \max_y \psi_n(y) \quad .$$

Let y_n denote a value of y such that

$$\psi_n = \psi_n(y_n) \quad .$$

It is seen from (7) below that such a value exists and is positive. Notation from Sections 1 and 2 will be continued.

First, we note the recursive relation

$$\psi_n(y) = \int_0^y tf(t) + \bar{F}(y)(y + \psi_{n-1}(y))$$

$$= \int_0^y \bar{F}(t)dt + \bar{F}(y)\ \psi_{n-1}(y) \quad , \quad y < 0 \quad .$$

Also, by considering the difference between having n and $n-1$ spares, we have

$$(6) \qquad \psi_n(y) - \psi_{n-1}(y) = \bar{F}^{n-1}(y)\{\int_0^y tf(t)dt + \bar{F}(y)(y + \mu) - \mu\}$$

$$= \bar{F}^{n-1}d(y)\{\int_0^y \bar{F}(t)dt - F(y)\mu\}$$

$$= \bar{F}^{n-1}(y)\{\phi_1(y) - \mu\} \quad .$$

On summing (6) we get

$$(7) \qquad \psi_n(y) - \psi_0(y) = \psi_n(y) - \mu$$

$$= \left(\int_0^y \bar{F}(t)dt - F(y)\mu\right)\left(\frac{1-\bar{F}^n(y)}{F(y)}\right)$$

$$= \left(\phi_1(y) - \mu\right)\left(\frac{1-\bar{F}^n(y)}{F(y)}\right) \quad .$$

<u>Proposition 7</u>: If $v_1 > v_0$, then $\psi_n > \psi_{n-1}$, $n \geq 1$. If $v = v_0 = \mu$ then $\psi_n = v_0$ for $n \geq 1$.

<u>Proof</u>: From (7),

$$\psi_n - \mu \geq \psi_n(x_1) - \mu$$

$$= \frac{1-\bar{F}^n(x_1)}{F(x_1)}(v_1 - \mu)$$

$$> 0 \quad .$$

That is, $\psi_n(y_n) > \mu$, $n \geq 1$, which from (7) implies

$$\phi_1(y_n) > \mu \quad , \quad n \geq 1 \quad .$$

Thus, from (6)

$$\psi_n - \psi_{n-1} \geq \psi_n(y_{n-1}) - \psi_{n-1}(y_{n-1})$$

$$= \bar{F}^{n-1}(y_{n-1})(\phi_1(y_{n-1}) - \mu)$$

$$> 0 \quad .$$

Suppose $v_1 = \mu$. The above argument shows that $\psi_n \geq \mu$. However, since $v_n \geq \psi_n$ and $v_n = \mu$, $n \geq 1$, by proposition 2, it follows that $\psi_n = \mu$, $n \geq 1$.

A reasonable conjecture is that for any set of maximizing values $\{y_n\}$, $y_{n+1} < y_n$, $n \geq 1$. We prove that this is, in fact, true.

First we need a well-known tool which will also be used in the next section. Let $g(\theta_i, y)$, $i = 0, 1$, be functions defined over a subset A of the real numbers. Assume $g(\theta_i, y)$ have maximums y_{θ_i} in A; i.e.,

$$g\left(\theta_i, y_{\theta_i}\right) = \max_{y \in A} g(\theta_i, y) \quad , \quad i = 0, 1 \quad .$$

<u>Lemma</u>: If for every $y', y'' \in A, y'' > y'$, we have

$$g(\theta_1, y'') - g(\theta_1, y') \leq g(\theta_0, y'') - g(\theta_0, y') \tag{8}$$

then $y_{\theta_1} \leq y_{\theta_0}$.

<u>Proof</u>: We have, by definition,

$$g(\theta_i, y) - g(\theta_i, y_{\theta_i}) \leq 0 \quad , \quad \forall\, y \in A \quad , \quad i = 0, 1.$$

Suppose $y > y_{\theta_0}$, $y \in A$. Then, by hypothesis, with $y = y''$, $y_{\theta_0} = y'$,

$$\begin{aligned} g(\theta_1, y) - g\left(\theta_1, y_{\theta_0}\right) &\leq g(\theta_0, y) - g\left(\theta_0, y_{\theta_0}\right) \\ &\leq 0 \quad , \end{aligned}$$

i.e.

$$g(\theta_1, y) \le g\left(\theta_1, y_{\theta_0}\right) \quad , \quad \forall\, y > y_{\theta_0} .$$

Therefore, $y_{\phi_1} \le y_{\phi_0}$.

We now show

<u>Proposition 8</u>: $y_{n+1} \le y_n$, $n \ge 1$.

<u>Proof</u>: Let $\theta_i = n + i$, $i = 0, 1$, so that in the lemma

$$g(\theta_i, y) = \log\, (\phi_1(y) - \mu)\ \frac{1-\bar{F}^{n+i}(y))}{F(y)}$$

$$= \log\, (\phi_1(y) - \mu) - \log F(y) + \log\, (1 - \bar{F}^{n+i}(y)) \quad ,$$

$i = 0, 1$.

Let A be the set of y for which $\phi_1(y) - \mu > 0$. Since, throughout, we are assuming $v_1 > v_0$, A is non-empty and, by (7) and Proposition 7, contains y_{n+i}, $i = 0, 1$. However, for every $y'' > y'(y', y'' \in A)$ we have

$$g(\theta_i, y'') - g(\theta_i, y') = \log \frac{1-\bar{F}^{n+i}(y'')}{1-\bar{F}^{n+i}(y')}, \quad i = 0, 1 \quad .$$

For (8) to hold it is sufficient that

$$\log \frac{1-\bar{F}^{n+1}(y'')}{1-\bar{F}^{n+1}(y')} \le \log \frac{1-\bar{F}^{n}(y'')}{1-\bar{F}^{n}(y')} \quad , \quad \forall \quad y'' > y' \quad ,$$

or equivalently,

$$\frac{1-\bar{F}^{n+1}(y'')}{1-\bar{F}^{n}(y')} \leq \frac{1-\bar{F}^{n+1}(y')}{1-\bar{F}^{n}(y'')} \quad , \quad \forall\, y'' > y' \quad .$$

This inequality will hold if

$$\frac{1-\bar{F}^{n+1}(y)}{1-\bar{F}^{n}(y)}$$

is non-increasing in y, or if

$$\frac{1-Z^{n+1}}{1-Z^{n}}$$

is increasing in Z for $0 < Z < 1$. However, for $n \geq 1$,

$$\begin{aligned} \frac{1-Z^{n+1}}{1-Z^{n}} &= \sum_{k=0}^{n} Z^k \Big/ \sum_{k=0}^{n-1} Z^k \\ &= 1 + Z^n \Big/ \sum_{k=0}^{n-1} Z^k \\ &= 1 + \Big(\sum_{k=1}^{n} \frac{1}{Z^k} \Big)^{-1} \quad , \end{aligned}$$

which is increasing in Z.

By Proposition 8 $\lim_{n\to\infty} \{y_n\}$ exists. We denote this limit by y^*.

The expression $\phi_1(y) - \mu$ represents the difference in the expected life if one spare is available compared to having none available when the replacement is made at time y. In our notation, $y_1 = x_1$ is the optimal time to make such a replacement. We have as a result of Proposition 8 a

<u>Corollary</u>: $\phi_1(y_n) - \mu$ is non-increasing in n.

Proof: Suppose for some $n \geq 1$ that

$$\phi_1(y_{n+1}) - \mu > \phi_1(y_n) - \mu \quad .$$

However, by Proposition 8, $y_{n+1} < y_n$. Since the other factor in (7) is decreasing in y, it would then follow that

$$\phi_n(y_{n+1}) - \mu > \phi_n(y_n) - \mu \quad ,$$

a contradiction.

4. Relationship Between Optimal and Optimal Equal Interval Schedules

Proposition 9:
$$\lim_{n\to\infty} \phi_n = \lim_{n\to\infty} v_n$$
$$= v \quad .$$

Proof: Consider the case of $x^* > 0$. From (1) and (2) and Proposition 5 on letting $n \to \infty$, we have

$$v = \int_0^{x^*} \bar{F}(t)dt + \bar{F}(x^*)v \quad ,$$

i.e.,

$$v = \int_0^{x^*} \frac{\bar{F}(t)dt}{F(x^*)} \quad .$$

However, from (7)

$$\lim_{n\to\infty} \phi_n(x) - \mu = \frac{1}{F(x)} \left\{\int_0^x \bar{F}(t)dt - F(x)\mu\right\}, \forall \quad x \quad .$$

Thus, in particular,

$$\lim_{n\to\infty} \phi_n \geq \lim_{n\to\infty} \phi_n(x^*)$$
$$= \int_0^{x^*} \bar{F}(t)dt/F(x^*)$$
$$= v \quad .$$

On the other hand, since $v_n \geq \phi_n$,

$$\lim_{n\to\infty} \phi_n \leq v \quad .$$

Now suppose $x^* = 0$. We have by (7) that

$$\lim_{n\to\infty} \psi_n \geq \int_0^x \bar{F}(t)dt/F(x) \quad , \quad \forall \quad x > 0 \quad ;$$

hence,

$$\lim_{n\to\infty} \psi_n \geq \lim_{x\to 0} \int_0^x \bar{F}(t)dt/F(x)$$

$$= 1/f(0) \quad .$$

If $\lim_{n\to\infty} v_n = \infty$, then, by Proposition 5, $f(0) = 0$. Thus, $\lim_{n\to\infty} \psi_n = \infty$. If $\lim_{n\to\infty} v_n < \infty$, by Proposition 4,

$$\begin{aligned} 1/v &= \lim_{n\to\infty} 1/v_{n-1} \\ &= \lim_{n\to\infty} r(x_n) \\ &= r(0) \\ &= f(0) \quad . \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \psi_n \geq v \quad .$$

Since $\lim_{n\to\infty} \psi_n \leq v_n = v$, the equality follows again.

Let

$$\begin{aligned} \psi(y) &= \lim_{n\to\infty} \psi_n(y) \\ &= \int_0^y \frac{\bar{F}(t)dt}{F(y)} \quad , \quad y > 0 \quad , \end{aligned}$$

and

$$\phi(y) = \lim_{n\to\infty} \phi_n(y)$$

$$= \int_0^y \bar{F}(t)dt + \bar{F}(y)v \quad , \quad y \geq 0 \quad .$$

Since $\psi_n(y) \uparrow \psi(y)$, $y > 0$, and $\psi_n(y)$ and $\psi(y)$ are continuous, then

$$\psi(y^*) = \max_{y>0} \psi(y)$$

$$= v \quad ,$$

if $y^* > 0$. If $v < \infty$, we have, for the same reasons, that

$$\phi(x^*) = \max_{y\geq 0} \phi(y)$$

$$= v \quad ,$$

implying that

$$\int_0^{x^*} \frac{\bar{F}(t)dt}{F(x^*)} = v \quad .$$

By applying the lemma of Section 3, with

$$g(\theta_0, y) = \log (\psi_n(y) - \mu)$$

and

$$g(\theta_1, y) = \log (\psi(y) - \mu) \quad ,$$

in the same manner as in the proof of Proposition 8 we can show that if $y^{**} > 0$ and $\psi(y^{**}) = \max_{y>0} \psi(y)$, then $y^{**} \in A$ and

$$y^{**} \leq y_n \quad , \quad \forall\, n \quad .$$

Thus,

$$y^{**} \leq y^* \quad .$$

Similarly, by letting

$$g(\theta_0, y) = \phi_n(y)$$

and

$$g(\theta_1, \mu) = \phi(y) \quad ,$$

if $v < \infty$ and $x^{**} \geq 0$ is such that

$$\phi(x^{**}) = \max_{y \geq 0} \phi(y) \quad ,$$

then

$$x^{**} \leq x_n \quad , \quad \forall\, n \quad ,$$

implying

$$x^{**} \leq x^* \quad .$$

Consequently, we can prove

<u>Proposition 10</u>: $x^* = y^*$.

<u>Proof</u>: If $x^* > 0$, $y^* > 0$, then by the preceding statements x^* and y^* are each the largest values that maximize $\psi(y)$ over $(0,\infty)$. Thus, $x^* = y^*$. We need only show that x^* and y^* are positive or zero together. Suppose $x^* > 0$. Then let $y^{**} = x^*$ in the remarks preceding Proposition 10 implying $y^* \geq y^{**} = x^* > 0$. Suppose $y^* > 0$. Then

$$1/f(0) \leq \psi(y^*) < \infty \quad .$$

If

$$\psi(y^*) > 1/f(0) \quad ,$$

then $x^* > 0$ since, by Proposition 4, $x^* = 0$ implies $v = 1/f(0)$. If

$$\psi(y^*) = 1/f(0) \quad ,$$

then $v = 1/f(0)$. However x^* is the largest value such that

$$\begin{aligned}\psi(x^*) &= \max_{y \geq 0} \phi(y) \\ &= 1/f(0) \quad .\end{aligned}$$

Therefore $x^* \geq y^* > 0$.

<u>Example</u>:

Uniform (0,1).

$$\begin{aligned}\psi_n(y) &= \frac{1}{2} + \frac{1-(1-y)^n}{y} \left(\frac{y(1-y)}{2}\right) \\ &= \frac{1}{2} \{1 + (1 - (1 - y)^n)(1 - y)\} \quad .\end{aligned}$$

Thus, y_n is that value of y that maximizes

$$(1 - y)(1 - (1 - y)^n) \quad .$$

Letting $Z = 1 - y$, we see that

$$Z_n = 1 - y_n = \left(\frac{1}{n+1}\right)^{1/n} \quad ,$$

or

$$y_n = 1 - \left(\frac{1}{n+1}\right)^{1/n} \quad , \quad n \geq 1 \quad ,$$

and

$$\psi_n = \frac{1}{2}\{1 + (1 - \frac{1}{n+1})(\frac{1}{n+1})^{1/n}\} \quad .$$

We can see directly, or using Proposition 9,

$$\lim_{n\to\infty} \psi_n = \lim_{n\to\infty} v_n$$
$$= 1 \quad .$$

Values of ψ_n and y_n are tabulated in Table 1.

5. The Expected Number of Spares Used

While there may be n spares available to extend the life of the system, because of possible failure of the system before all spares are used, the actual number used is a random variable depending on F and the schedule.

Let $S = \{Z_n, Z_{n-1}, \ldots, Z_1, T\}$ denote the schedule. Let u_n denote the expected number of spares used. Then, u_n satisfies the recursive relationship

$$(9) \qquad u_n = \bar{F}(Z_n)(1 + u_{n-1}) \quad , \quad n \geq 1$$

$$u_0 = 0 \quad .$$

In case S is the equal interval schedule with interval length y, then

$$(10) \qquad \begin{aligned} u_n &= \bar{F}(y) + \bar{F}(y)u_{n-1} \\ &= \sum_{k=1}^{n} \bar{F}^k(y) \\ &= \frac{\bar{F}(y) - \bar{F}^{n+1}(y)}{F(y)} \quad , \end{aligned}$$

which can also be obtained from the geometric distribution.

Examples

(a) Uniform (0,1)

If the optimal schedule is used then (9) yields

$$u_n = (1 - x_n) + (1 - x_n)\, u_{n-1}$$
$$= v_{n-1}(1 + u_{n-1}) \quad .$$

If the optimal fixed interval schedule is used, then (10) yields

$$u_n = \frac{\left(\frac{1}{n+1}\right)^{1/n} - \left(\frac{1}{n+1}\right)^{\frac{n+1}{n}}}{(1-1/n+1)^{1/n}}$$

$$= \frac{\left(\frac{1}{n+1}\right)^{1/n}\left(1-\left(\frac{1}{n+1}\right)\right)}{1-\left(1/n+1\right)^{1/n}}$$

$$= \frac{\left(\frac{1}{n+1}\right)^{1/n} \frac{n}{n+1}}{1-\left(1/n+1\right)^{1/n}} \quad .$$

These values will be tabulated in Table 1.

(b) Two exponentials in parallel.

In this case, the formula (9) for the optimal schedule yields,

$$u_n = (1 - (2v_{n-1} - 1)^{-2})(1 + u_{n-1}) \quad , \quad u_0 = 0 \quad .$$

These values will be tabulated in Table 2. We shall be especially interested in comparing these values with expected number of components used in the situations described in Sections 6 and 7.

6. Systems With More Than One Replaceable Component

In the previous sections it was assumed that when a replacement was scheduled, the entire system was replaced. However, example (b) arose in the context of the system consisting of a two independent component module, each component having the same exponential life distribution. This suggests that efficiencies can be gained by exploiting the structure of the system and the individual life distributions of the components.

We concentrate, here, on the specific structure of example (b) and consider schedules that call for the replacement of individual components. A number, n, of spare components (as distinguished from spare systems) are on hand. The objective is, still, to attempt to maximize the expected life of the system.

To put the problem in a general context, a working system can be thought of as being in state (s,t), meaning that s time units have elapsed since it was known, either by replacement or deduction, that component 1 was in working order. The same meaning is given to t for component 2. The deduction that component 1 is working is made when the inspection (which we assume takes place) of component 2 after it is replaced reveals that it has failed, implying that component 1 is working, and, because of its exponential character, is as good as a new component. In state (s,t), at least one of the components is working.

Remark: For the above to be applicable we are assuming that:

a) inspection of a component is destructive (for otherwise, one would continue to use a working component rather than replace it),

b) inspection is informative in the sense that it tells us whether or not the component was functioning immediately before inspection.

Let $w(s,t)$ denote the expected life of the system when, initially, the state is (s,t), n spares are available, and an optimal replacement schedule is used.

Special Case: Assume $(s,t) = (0,0)$ and $n = 1$, and assume that component 1 is optimally replaced at time Z_1. Then

$$w_1(0,0) = \max_Z \{2 \int_0^Z ye^{-y}(1-e^{-y})dy + e^{-2Z}(Z + 3/2) + (1-e^{-Z})e^{-Z}(Z + 3/2)$$

$$+ e^{-Z}(1 - e^{-Z})(Z + 1)\}$$

$$= 2 \int_0^{Z_1} ye^{-y}(1 - e^{-y})dy + (5/2 + Z_1)e^{-Z_1} - (1 + Z_1)e^{-2Z_1} \quad .$$

It is seen, in a straight forward manner, that

$$e^{-Z_1} = 1/2$$

or

$$Z_1 = \log 2 \quad ,$$

from which it follows that

$$w_1(0,\ 0) = 13/8 \quad .$$

The Case of $n \geq 1$.

When $n \geq 1$, the problem of obtaining an optimal component replacement schedule appears to be difficult to solve. For the purpose of, at least, achieving lower bounds on what could be accomplished by an optimal schedule, we restrict ourselves to a subclass of schedules for which the analysis is feasible.

We assume, initially, that $(s,t) = (0,0)$ and consider schedules of the following form: when $n \geq 2$, at time $\tilde{x}_n$, component 1 is replaced and then inspected. If the inspection reveals that the component was working, then component 2 is also replaced. If the inspection shows that the component was not working, then component 2 is left intact. When $n = 1$, component 1 is replaced at time Z_1.

The rationale for the type of rule is based on the exponential life distribution of the components, the parallel structure of the system, and mathematical convenience. If, when the system is working, one component is not functioning, then the other must be, and, since it is working it must be, statistically speaking, equivalent to a new component. Both components are replaced when the one inspected is functioning so that the state is always $(0,0)$ after a scheduled replacement. However, the double replacement is not completely inefficient, for if the one inspected is working there is an enhanced possibility (depending, of course, on $\tilde{x}_n$) that the other will have failed.

Let $\tilde{v}_n$ denote the maximal life of the system when n spare components are available and the class of schedules is restricted to the above stated class. Then $\tilde{v}_n$ satisfies

$$\tilde{v}_n = \max_x \{\eta_n(x)\} \quad , \quad n \geq 2 \quad ,$$

$$\tilde{v}_1 = w_1(0,0)$$

$$= 13/8 \quad ,$$

$$\tilde{v}_0 = 3/2 \quad ,$$

where, for $n \geq 2$,

$$(11) \qquad \eta_n(x) = 2 \int_0^x te^{-t}(1 - e^{-t})dt$$

$$+ \left(1 - (1 - e^{-x})^2\right)\left\{x + \frac{(1-e^{-x})e^{-x}}{1-(1-e^{-x})^2}\tilde{v}_{n-1} + \frac{e^{-x}}{(1-(1-e^{-x})^2)}v_{n-2}\right\}$$

$$= 2 \int_0^x te^{-t}(1 - e^{-t})dt + \left(1 - (1 - e^{-x})^2\right)x + e^{-x}(1 - e^{-x})\tilde{v}_{n-1}$$

$$+ e^{-x}\tilde{v}_{n-2} \quad .$$

The derivative of $\eta_n(x)$ is

$$\eta_n'(x) = e^{-x}\{2\tilde{v}_{n-1} - 1)e^{-x} + 2 - \tilde{v}_{n-1} - \tilde{v}_{n-2}\} \quad .$$

By inspecting $\eta_n'(x)$ it is clear that

$$e^{-\tilde{x}_n} = \frac{\tilde{v}_{n-1}+\tilde{v}_{n-2}-2}{2\tilde{v}_{n-1}-1}$$

or

$$\tilde{x}_n = \log \frac{2\tilde{v}_{n-1}-1}{\tilde{v}_{n-1}+\tilde{v}_{n-2}-2} \quad , \quad n \leq 2 \quad .$$

If $n = 1$, then $\tilde{x}_1 = x_1 = \log 2$.

On Substituting $\tilde{x}_n$ in (11) we get, after some calculations, that

$$\tilde{v}_n = \frac{1}{2}\{3 + \frac{(\tilde{v}_{n-1}+\tilde{v}_{n-2}-2)^2}{2\tilde{v}_{n-1}-1}\} \quad , \quad n \geq 2 \quad .$$

To calculate $\tilde{u}_n$, the expected number of spare components used in the life of the system, we have

$$\tilde{u}_0 = 0$$

$$\tilde{u}_1 = 1 - (1 - e^{-\tilde{x}_1})^2$$

$$= 3/4$$

$$\tilde{u}_n = (1 - e^{-\tilde{x}_n})e^{-\tilde{x}_n}(1 + \tilde{u}_{n-1}) + e^{-\tilde{x}_n}(2 + \tilde{u}_{n-2}) \quad , \quad n \geq 2 \quad .$$

The formulas for $\tilde{x}_n$, $\tilde{v}_n$, and $\tilde{u}_n$ are recursive and are readily calculated numerically. (See Table 3.) In comparing x_n, v_n and u_n with $\tilde{x}_n$, $\tilde{v}_n$ and $\tilde{u}_n$ it should be kept in mind that n, in the former case, because the entire system is replaced is, in fact, twice the number of components of the latter case.

7. The Option of Non-destructive Inspections

In Section 6 it was assumed, presumably due to destructive inspections, that an inspection implied a replacement. Here we assume that inspections are not destructive so that if a component is found to be working, it is continued in use. Without assuming inspection costs, continuous surveillance would be optimal leading to an expected system life equal to $\frac{n+1}{2} + 1$. Instead of introducing inspection costs into the model we shall derive formulas for computing the values of the characteristics assuming the schedule derived in Section 6. In this way we can compare destructive and non-destructive inspections with respect to a fixed schedule.

Let u_n^* denote the expected number of replacements made during the life of the system. Then

$$u_1^* = \left(1 - e^{-\tilde{x}_1}\right)^2 \cdot 0 + e^{-2\tilde{x}_1} u_1^* + 2e^{-\tilde{x}_1}\left(1 - e^{-\tilde{x}_1}\right)$$

i.e.,

$$u_1^* = \frac{2e^{-\tilde{x}_1}(1-e^{-\tilde{x}_1})}{1-e^{-2\tilde{x}_1}}$$

$$= 2/3 \quad .$$

For $n \geq 2$,

$$u_n^* = \left(1 - e^{\tilde{x}_n}\right)^2 \cdot 0 + e^{-2\tilde{x}_n} u_n^* + 2e^{-\tilde{x}_n}\left(1 - e^{-\tilde{x}_n}\right)\left(1 + u_{n-1}^*\right)$$

i.e.,

$$u_n^* = \frac{1}{1-e^{-2\tilde{x}_n}} \{2e^{-\tilde{x}_n} (1 - e^{-\tilde{x}_n})(1 + u_{n-1}^*)\} \quad .$$

Let v_n^* denote the expected life of the system. Then

$$v_0^* = 3/2$$

$$v_n^* = 2 \int_0^{\tilde{x}_n} te^{-t}(1 - e^{-t})dt + (1 - (1 - e^{-\tilde{x}_n})^2)\, \tilde{x}_n$$

$$+ e^{-2\tilde{x}_n} v_n^* + 2e^{-\tilde{x}_n}(1 - e^{-\tilde{x}_n})\, v_{n-1}^* \quad , \quad n \geq 1 \quad .$$

i.e., for $n \geq 1$,

$$v_n^* = \frac{1}{1-e^{-2\tilde{x}_n}} \{ \int_0^{\tilde{x}_n} (1 - (1 - e^{-t})^2)dt + 2e^{-\tilde{x}_n}(1 - e^{-\tilde{x}_n})v^*_{n-1}\}$$

$$= \frac{1}{1-2^{-2\tilde{x}_n}} \{\frac{3}{2} - 2e^{-\tilde{x}_n} + \frac{e^{-2\tilde{x}_n}}{2} + 2e^{-\tilde{x}_n}(1 - e^{-\tilde{x}_n})\, v_{n-1}^*\} \quad .$$

Values of v_n^* and n_n^* can be recursively calculated and are tabulated in Table 3.

Considering that n in Table 2 is, in fact, twice the value of n in Table 3, replacement of components rather than systems produces a substantial gain in expected life for a given number of spare components. On the other hand, in comparing the two parts of Table 3, non-destructive rather than destructive inspection produces only a modest gain.

8. Asymptotic Distribution of System Life

Let $\bar{P}_n(t)$ denote the probability that system life exceeds t when n spares are available and an optimal replacement schedule is used. Let $\tilde{P}_n(t)$ denote the same probability when an optimal equal interval replacement schedule is used. When $x^* = y^* = 0$ and $v = \psi = 1/f(0) > 0$, it would seem that the limiting distribution as $n \to \infty$ of system life should be exponential with rate $f(0)$. That this is true depends on the following.

Proposition 11:[2]

$$\sum_{i=1}^{\infty} x_i = \lim_{n\to\infty} ny_n = \infty \quad .$$

Proof: In the cases where $x^* = y^* > 0$, the proposition is trivially true. Also, since

$$v_n \leq \sum_{j=1}^{n} x_j + \mu$$

and

$$\psi_n \leq ny_n + \mu \quad ,$$

the proposition is immediate when $v = \psi = \infty$. Thus, consider the remaining case where $v = \psi = 1/f(0) < \infty$ and $x^* = y^* = 0$. Since the

[2] We are indebted to D. R. Smith for some remarks which were very helpful in establishing this proposition.

sequences $\{x_n\}$ and $\{y_n\}$ are non-increasing and because f is continuous of $x = 0$, for any $\varepsilon > 0$ there exists an N such that for $n > N$

(12) $$F(x_n) \leq (f(0) + \varepsilon)x_n, \; F(y_n) \leq (f(0) + \varepsilon)y_n \quad .$$

On iterating (5)

$$v_n - v_{n-1} \geq \prod_{j=1}^{n-1} \bar{F}(x_j)(v_1 - v_0) \quad , \quad \forall\, n \geq 2 \quad .$$

Since $v < \infty$, $\lim_{n\to\infty} (v_n - v_{n-1}) = 0$ implying

$$\prod_{j=1}^{\infty} F(x_j) = 0 \quad ,$$

which, with (12), implies $\sum_{j=1}^{\infty} x_j = \infty$. From (7), with $y = y_n$, on letting $n \to \infty$ one has that

$$\lim_{n\to\infty} \bar{F}^n(y_n) = 0 \quad ,$$

which, with (12), implies $\lim_{n\to\infty} ny_n = \infty$.

We now have

<u>Proposition 12</u>: If $x^* = 0$, then for every $t > 0$,

$$\begin{aligned} \lim_{n\to\infty} \bar{P}_n(t) &= \lim_{n\to\infty} \tilde{P}_n(t) \\ &= e^{-f(0)t} \quad , \quad \text{if } f(0) > 0 \\ &= 1 \quad , \quad \text{if } f(0) = 0 \quad . \end{aligned}$$

Proof: Fix t. From Proposition 11, $\sum_{j=1}^{\infty} x_j = \infty$. Hence, for every k there exists an N such that for $n > N$, $\sum_{j=k}^{n} x_j > t$. Thus, since the x_n's are decreasing the component in use at time t or before is never older than x_k. Denote by $r_n(t)$, the failure rate function for the system life. Then

$$\min_{0 \le s \le x_k} r(s) \le r_n(\tau) \le \max_{0 \le s \le x_k} r(s) \quad , \quad 0 \le \tau \le t \quad .$$

Thus, using

$$\bar{P}_n(t) = e^{-\int_0^t r_n(\tau)d\tau} \quad ,$$

$$e^{-t \max_{0 \le s \le x_k} r(s)} \le \bar{P}_n(t) \le e^{-t \min_{0 \le s \le x_k} r(s)} \quad , \quad \forall\, k \quad \text{and} \quad n > N \quad .$$

Letting $n \to \infty$ and then $k \to \infty$ and since $x_k \to 0$, we have the value for $\lim_{n \to \infty} \bar{P}_n(t)$. The value for $\lim_{n \to \infty} \tilde{P}_n(t)$ is proved similarly.

9. A Multi-Failure Generalization

Let us now suppose that it takes i $(i \geq 1)$ vital component failures to cause a system failure. Component replacement occurs at a component failure or when scheduled. In previous sections $i = 1$. As before we suppose there are n, $n \geq i-1$, spares and we let $v(n,i)$ denote the expected life of the system under an optimal schedule when the system fails at the moment of the i^{th} vital component failure and n spares are available. Then $v(n,i)$ satisfies the optimality equation

$$v(n,i) = \max_{0 \leq x \leq T} \left\{ \int_0^x \bar{F}(t)dt + \bar{F}(x)v(n-1,i) + F(x)v(n-1,i-1) \right\}, n \geq i, i \geq 1,$$

$$v(i-1,i) = i\mu \quad \text{(Replacement only occurs at component failures when } n = i-1.\text{)} ,$$

$$v(n, 0) = 0 \quad , \quad n \geq 0 \quad .$$

Let us denote by $x(n, i)$ the largest value that maximizes the above expression within brackets. The following proposition whose proof by induction on $k \equiv i + n$ will not be given as it is almost identical with the proof of Lemma 3 of [1].

Proposition 13:

(a) $v(n + 1, i + 1) - v(n + 1, i) \geq v(n, i + 1) - v(n, i)$

(b) $v(n + 1, i) - v(n, i) \geq v(n + 2, i) - v(n + 1, i)$

(c) $v(n,i+1) - v(n,i) \geq v(n,i+2) - v(n,i+1)$.

From the lemma of Section 3 and the above proposition we have the following.

Corollary: $x(n,i)$ is decreasing in n and increasing in i.

In addition, it follows analogous to the results of Section 2 that $x(n, i) = T$ whenever F is NWUE and

$$r(x(n,i) = \frac{1}{v(n-1,i)-v(n-1,i-1)} \quad , \quad n \geq i \quad , \quad i \geq 1 \quad .$$

10. Further Remarks

Proposition 3 and 13(b) are relevant to the question of how many spare parts should be stocked. The propositions assert that the incremental gain in expected life is decreasing in n. Thus, if the reward associated with a unit of expected life can be compared to the cost of a spare, there is a determinable N such that it is not economically efficient to shock $n > N$ spares. Although it seems a reasonable conjecture, we have not been able to show that $\psi_n - \psi_{n-1} \downarrow$ in n.

Suppose, instead of maximizing expected life, it is more relevant to maximize the probability that the system survives t units of time. Then, in the case of $i = 1$, if the intervals between scheduled replacements of n spares is $\{z_n, z_{n-1}, \ldots, z_1\}$, then

$$P\{L > t\} = \prod_{i=0}^{n} \bar{F}(z_i) \quad ,$$

where

$$\sum_{i=0}^{n} z_i = t \quad , \quad z_i \geq 0, \; i = 0, \ldots, n \quad , \tag{13}$$

presuming it is always optimal to schedule the use of all spares. Maximizing $P\{L > t\}$ subject to (13) is equivalent to maximizing $\sum_{i=0}^{n} \log \bar{F}(z_i)$ subject to (13), a classical allocation of resources problem. If F is IFR, then $\log \bar{F}(z)$ is concave and it is well-known that the optimal values of $z_0, \ldots, z_n$ are equal; i.e., $z_k = t/n+1$, $k = 1, \ldots, n$. If F is DFR then $\log \bar{F}(z)$ is convex and an optimal schedule is to have $z_n = t$, $z_k = 0$, $k = 1, \ldots, n-1$.

TABLE 1. UNIFORM (0,1) DISTRIBUTION

n	v	x	u	ψ	y	u
0	.5000	-----	.0000	.5000	-----	.0000
1	.6250	.5000	.5000	.6250	.5000	.5000
2	.6953	.3750	.9375	.6924	.4226	.9106
3	.7417	.3046	1.3471	.7362	.3700	1.2768
4	.7750	.2582	1.7409	.7674	.3312	1.6150
5	.8003	.2249	2.1244	.7911	.3011	1.9336
6	.8203	.1996	2.5007	.8098	.2769	2.2374
7	.8364	.1796	2.8716	.8250	.2570	2.5296
8	.8498	.1635	3.2384	.8377	.2401	2.8122
9	.8610	.1501	3.6019	.8484	.2257	3.0869
10	.8707	.1389	3.9627	.8576	.2132	3.3548
11	.8790	.1292	4.3212	.8656	.2022	3.6167
12	.8864	.1209	4.6779	.8727	.1924	3.8734
13	.8928	.1135	5.0329	.8789	.1837	4.1255
14	.8985	.1071	5.3865	.8845	.1758	4.3734
15	.9037	.1014	5.7389	.8896	.1687	4.6176
16	.9083	.0962	6.0902	.8942	.1622	4.8583
17	.9125	.0916	6.4405	.8983	.1563	5.0959
18	.9163	.0874	6.7900	.9022	.1509	5.3306
19	.9198	.0836	7.1387	.9057	.1458	5.5627
20	.9230	.0801	7.4867	.9089	.1412	5.7922
30	.9449	.0566	10.9384	.9315	.1081	7.9797
40	.9570	.0438	14.3581	.9445	.0886	10.0283
50	.9647	.0358	17.7597	.9531	.0756	11.9836
60	.9701	.0303	21.1494	.9592	.0662	13.8699
70	.9740	.0262	24.5309	.9638	.0590	15.7023
80	.9770	.0231	27.9062	.9674	.0534	17.4907
90	.9794	.0207	31.2767	.9703	.0488	19.2422
100	.9813	.0187	34.6435	.9727	.0451	20.9621

TABLE 2. $f(t) = 2e^{-t}(1-e^{-t})$

n	v	x	u
0	1.5000	-----	.0000
1	1.7500	1.0986	.7500
2	1.9500	.8472	1.4700
3	2.1224	.7191	2.1763
4	2.2765	.6370	2.8746
5	2.4172	.5785	3.5676
6	2.5476	.5339	4.2570
7	2.6697	.4984	4.9435
8	2.7849	.4693	5.6279
9	2.8943	.4448	6.3105
10	2.9987	.4238	6.9917
11	3.0988	.4056	7.6717
12	3.1950	.3896	8.3507
13	3.2877	.3754	9.0289
14	3.3774	.3626	9.7063
15	3.4643	.3510	10.3830
16	3.5486	.3405	11.0592
17	3.6306	.3310	11.7348
18	3.7105	.3221	12.4100
19	3.7883	.3140	13.0847
20	3.8644	.3064	13.7591
30	4.5454	.2524	20.4878
40	5.1259	.2196	27.1994
50	5.6406	.1970	33.9015
60	6.1078	.1802	40.5972
70	6.5387	.1671	47.2887
80	6.9407	.1565	53.9769
90	7.3188	.1477	60.6626
100	7.6770	.1402	67.3464

TABLE 3.

n	$\tilde{x}$	$\tilde{v}$	$\tilde{u}$	v^*	u^*
0	-----	1.5000	.0000	1.5000	------
1	.6931	1.6250	.7500	1.8333	.6666
2	.6931	1.7812	1.4375	1.9166	1.1666
3	.6000	1.8858	2.1127	2.0536	1.7273
4	.5083	2.0013	2.8136	2.1654	2.3381
5	.4644	2.0930	3.4753	2.2729	2.9437
6	.4195	2.1883	4.1723	2.3670	3.5500
7	.3921	2.2707	4.8328	2.4583	4.1710
8	.3647	2.3537	5.5237	2.5415	4.7807
9	.3454	2.4289	6.1858	2.6222	5.4079
10	.3266	2.5035	6.8714	2.6975	6.0224
11	.3122	2.5730	7.5353	2.7707	6.6524
12	.2983	2.6415	8.2169	2.8401	7.2715
13	.2869	2.7062	8.8822	2.9075	7.9033
14	.2761	2.7699	9.5607	2.9721	8.5263
15	.2669	2.8309	10.2272	3.0351	9.1594
16	.2582	2.8907	10.9033	3.0958	9.7856
17	.2505	2.9483	11.5706	3.1550	10.4197
18	.2433	3.0049	12.2450	3.2124	11.0486
19	.2368	3.0597	12.9128	3.2685	11.6837
20	.2307	3.1135	13.5859	3.3231	12.3149
30	.1878	3.5950	20.2823	3.8116	18.6817
40	.1623	4.0056	26.9702	4.2266	25.0900
50	.1449	4.3699	33.6533	4.5938	31.5261
60	.1321	4.7005	40.3332	4.9267	37.9823
70	.1221	5.0055	47.0109	5.2334	44.4542
80	.1141	5.2900	53.6870	5.5193	50.9386
90	.1075	5.5577	60.3619	5.7881	57.4334
100	.1019	5.8112	67.0359	6.0426	63.9370

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